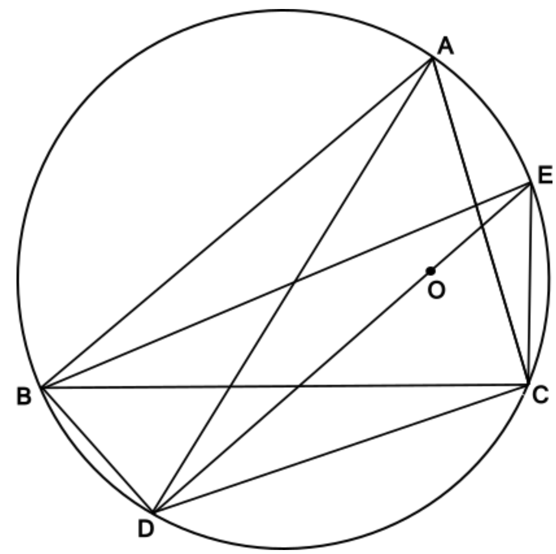
In the above picture, ΔABC is inscribed in the circle. O is its orthocentre and AD is the diameter of the circle. DO is produced to meet the other side of the circle at E.

Prove: $BE^2 + EC^2 + CD^2 + DB^2 = 2DE^2$.



Question framed by
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<u>**Author's Solution**</u>

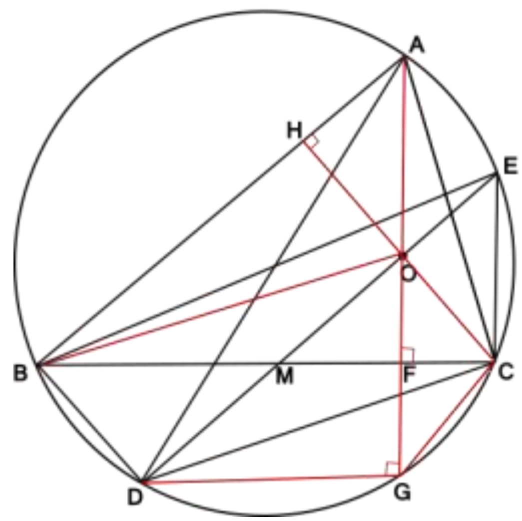
Given :

ΔABC is inscribed in the circle. O is its orthocentre and AD is the diameter. DO produced meets the circle at E.

To Prove :

$\boldsymbol{BE^2 + EC^2 + CD^2 + DB^2 = 2DE^2}$.

Construction:



Produce AO to meet BC and the circle at F & G respectively. Produce CO to meet AB at H. Join BO & DG. Let DE & BC meet at M.

Solution :

BOCD is a parallelogram.

This can be proved in many ways. Here, the following method is employed.

$\angle HAO = \angle FCO$ ---------------------------------- (1)

[O is orthocentre. AF$\perp$ BC & CH $\perp AB$ and AHFC is concyclic]

$\angle BAG = \angle BCG$ [BACG is concyclic]

$i.e., \angle HAO = \angle FCG$ --------------------- (2)

(1) & (2) →

$\angle FCO = \angle FCG$

∴ FC is perpendicular bisector of OG.

And OC = CG -------------------------------- (3)

AD is diameter

∴ ∠AGD = 90°

∴ BC ∥ DG

And BCGD is an isosceles trapezium. (Any trapezium with concyclic vertices is an isosceles trapezium)

∴ CG = BD ------------------------------------- (4)

(3) & (4) → OC = BD -------------------------(5)

CH ∥ BD [CH is altitude and $\angle ABD$ is right angle]

⇒ CO ∥ BD ------------------------------- (6)

(5) & (6) →

BOCD is parallelogram.

∴ M is the midpoint of BC, the diagonal.

⇒ EM & DM are medians of Δ BEC & Δ BDC respectively.

∴ As per Apollonius Theorem,

$BE^2 + EC^2 = 2\,(EM^2 + BM^2)$ ------------------------ (7)

And $CD^2 + DB^2 = 2\,(DM^2 + BM^2)$ ------------------------ (8)

(7) + (8) →

$BE^2 + EC^2 + CD^2 + DB^2 = 2\,(EM^2 + DM^2 + 2BM^2)$

$= 2\,[EM^2 + DM^2 + 2(EM \times DM)]$

$= 2\,(EM + DM)^2$

$BE^2 + EC^2 + CD^2 + DB^2 = 2\,DE^2$ ----------------------------- Proved

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